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Low-energy limit of chiral meson theory

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Abstract. Based on a phenomenologically successful effective chiral theory of pseudoscalar, vector, and axial-vector mesons, all the coefficients of the chiral perturbation theory are predicted. There is no new adjustable parameter in these predictions. Up to $O(m_q^2)$ the formulas of the masses of the pseudoscalar mesons are the same as the ones obtained by ChPT.

PACS. 11.30.Rd Chiral symmetries – 13.75.Lb Meson-meson interactions – 12.40.-y Other models for strong interactions

The chiral perturbation theory(ChPT) proposed a decade ago [1] is successful in parametrizing low-energy QCD. The chiral symmetry revealed from QCD, quark mass expansion, and momentum expansion are used to construct the Lagrangian of ChPT

$$
\begin{aligned}
\mathcal{L} = &\frac{f_\pi^2}{16}\mathrm{Tr}D_\mu U D^\mu U^\dagger + \frac{f_\pi^2}{16}\mathrm{Tr}\chi(U+U^\dagger)\\
&+L_1[\mathrm{Tr}(D_\mu U D^\mu U^\dagger)]^2 + L_2(\mathrm{Tr}D_\mu U D_\nu U^\dagger)^2\\
&+L_3\mathrm{Tr}(D_\mu U D^\mu U^\dagger)^2 + L_4\mathrm{Tr}(D_\mu U D^\mu U^\dagger)\mathrm{Tr}\chi(U+U^\dagger)\\
&+L_5\mathrm{Tr}D_\mu U D^\mu U^\dagger(\chi U^\dagger + U\chi) + L_6[\mathrm{Tr}\chi(U+U^\dagger)]^2\\
&+L_7[\mathrm{Tr}\chi(U-U^\dagger)]^2 + L_8\mathrm{Tr}(\chi U\chi U + \chi U^\dagger\chi U^\dagger)\\
&-iL_9\mathrm{Tr}(F^L_{\mu\nu}D^\mu U D^\nu U^\dagger + F^R_{\mu\nu}D^\mu U^\dagger D^\nu U)\\
&+L_{10}\mathrm{Tr}(F^L_{\mu\nu}UF^{\mu\nu R}U^\dagger). \qquad (1)
\end{aligned}
$$

The parameters in the chiral perturbation theory(1) are pion decay constant f_π and the 10 Gasser-Leutwyler(G-L) coefficients. These parameters are determined by fitting experimental data (table 1). Recent review of the ChPT can be found in ref. [9]. Many models [2–8] try to predict some of the 10 coefficients (table 2).

The chiral perturbation theory is rigorous and phenomenologically successful in describing the physics of the pseudoscalar mesons at low energies. Models attempt to deal with the two main frustrations that the ChPT is limited to pseudoscalar mesons at low-energy and contains many coupling constants which must be measured. However, the chiral perturbation theory sets a low-energy limit for all models. The price paid for including more mesons and determining the coefficients is to make additional assumption.

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I have proposed an effective chiral theory of pseudoscalar, vector, and axial-vector mesons [10, 11]. It provides a unified description of low-lying meson physics. This effective theory is phenomenologically successful [10–12]. It is natural to study whether the Lagrangian (1) of the chiral perturbation theory can be derived from this effective theory and the 10 coefficients can be predicted. In ref. [13] in terms of this theory [10,11] an analytic method has been used to study $\mathcal{L}_{1,2,3,9,10}$. In this paper the effective theory [10, 11] has been used to predict all the 10 coefficients of ChPT.

The Lagrangian of this theory has been constructed as

$$
\begin{aligned}
\mathcal{L} = &\bar\psi(x)(i\gamma\cdot\partial + \gamma\cdot v + \gamma\cdot a\gamma_5 - mu(x))\psi(x)\\
&-\bar{\psi(x)}M\psi(x) + \frac{1}{2}m_0^2(\rho_i^\mu\rho_{\mu i} + K^{*\mu}K^*_\mu + \omega^\mu\omega_\mu + \phi^\mu\phi_\mu\\
&+a_i^\mu a_{\mu i} + K_1^\mu K_{1\mu} + f^\mu f_\mu + f_{1s}^\mu f_{1s\mu}), \qquad (2)
\end{aligned}
$$

where M is the quark mass matrix

$$
\begin{pmatrix} m_u & 0 & 0\\ 0 & m_d & 0\\ 0 & 0 & m_s \end{pmatrix},
$$

$v_\mu = \tau_i\rho^i_\mu + \lambda_a K^{*a}_\mu + (\frac{2}{3} + \frac{1}{\sqrt{3}}\lambda_8)\omega_\mu + (\frac{1}{3} - \frac{1}{\sqrt{3}}\lambda_8)\phi_\mu$,
$a_\mu = \tau_i a^i_\mu + \lambda_a K^a_{1\mu} + (\frac{2}{3} + \frac{1}{\sqrt{3}}\lambda_8)f_\mu + (\frac{1}{3} - \frac{1}{\sqrt{3}}\lambda_8)f_{1s\mu}$,
and $u = \exp\{i\gamma_5(\tau_i\pi_i + \lambda_a K_a + \lambda_8\eta_8 + \eta_0)\}$. u is expressed as

$$
u = \frac{1}{2}(1+\gamma_5)U + \frac{1}{2}(1-\gamma_5)U^\dagger, \qquad (3)
$$

where $U = \exp\{i(\tau_i\pi_i + \lambda_a K_a + \lambda_8\eta + \eta_0)\}$.

The kinetic terms of mesons are generated by quark loops. Equation (2) is a Lagrangian of an effective theory,

Table 1. Values of the coefficients [8,14]

$10^3 L_1$	$10^3 L_2$	$10^3 L_3$	$10^3 L_4$	$10^3 L_5$	$10^3 L_6$	$10^3 L_7$	$10^3 L_8$	$10^3 L_9$	$10^3 L_{10}$
$.4 \pm .3$	$1.35 \pm .3$	-3.5 ± 1.1	$-.3 \pm .5$	$1.4 \pm .5$	$-.2 \pm .3$	$-.4 \pm .2$	$.9 \pm .3$	$6.9 \pm .7$	$-5.5 \pm .7$
$.52 \pm .23$	$.72 \pm .24$	$-2.70 \pm .99$		$.65 \pm .12$		$-.26 \pm .15$	$.47 \pm .18$		

Table 2. Coefficients obtained by models

	Vectors [4]	Quark [2]	ref. [6]	ref. [7]	Nucleon loop	Linearσ model	ENJL [8]
$(L_1+\frac{1}{2}L_3)\times 10^{-3}$	-2.1	$-.8$	2.1	1.1	$-.8$	$-.5$	
$L_1 \times 10^{-3}$							0.8
$L_2 \times 10^{-3}$	2.1	1.6	1.6	1.8	.8	1.5	1.6
$L_3 \times 10^{-3}$							-4.1
$L_4 \times 10^{-3}$							0.
$L_5 \times 10^{-3}$							1.5
$L_6 \times 10^{-3}$							0.
$L_7 \times 10^{-3}$							
$L_8 \times 10^{-3}$							0.8
$L_9 \times 10^{-3}$	7.3	6.3	6.7	6.1	3.3	.9	6.7
$L_{10} \times 10^{-3}$	-5.8	-3.2	-5.8	-5.2	-1.7	-2.0	-5.5

therefore, a cut-off is necessary to be introduced. In the chiral limit in ref. [10] under the cut-off it is defined

$$\frac{m^2 N_C}{(2\pi)^4}\frac{D}{4}\int \mathrm{d}^D k \frac{-i}{(k^2-m^2)^2} = \frac{F^2}{16}.$$

The vector field, for example, the ρ field of eq. (2) is normalized to physical ρ-meson field

$$\rho^i_\mu \rightarrow \frac{1}{g}\rho^i_\mu,$$

where g is a universal coupling constant and defined as

$$g^2 = \frac{1}{6}\frac{F^2}{m^2}.$$

In order to cancel the mixing between a^i_μ and pion fields a transformation

$$a^i_\mu \rightarrow \frac{1}{g_a}a^i_\mu - \frac{c}{g}\partial\pi^i$$

is introduced, where

$$g_a = \frac{1}{g}(1-\frac{1}{2\pi^2 g^2})^{-1}$$

and c is determined to be

$$c = \frac{f^2_\pi}{2gm^2_\rho}.$$

Physical pion field is defined as

$$\pi^i \rightarrow \frac{2}{f_\pi}\pi^i,$$

where

$$f^2_\pi = F^2(1-\frac{2c}{g}).$$

There are similar equations for other meson fields. The universal coupling constant g appears in all kinds of meson vertices. In ref. [10] the coupling between photon and ρ-meson (VMD) has been determined to be

$$\frac{e}{2}g\{-\frac{1}{2}F^{\mu\nu}(\partial_\mu\rho^0_\nu - \partial_\nu\rho^0_\mu) + A^\mu j^0_\mu\},$$

where j^0_μ is the meson vector current. Using this vertex, the numerical value of g is determined by the decay rate of $\rho \rightarrow ee^+$ to be 0.39. It is useful to summarize the features and results of this effective theory [10,11].

1. In the limit $m_q \rightarrow 0$, the theory is explicit chiral symmetric,
2. The scheme of the nonlinear σ-model is used to introduce the pseudoscalar mesons into eq. (2). In this way, the parameter m is naturally introduced to the theory and nonzero quark condensate is revealed from this theory. The theory has dynamical chiral symmetry breaking,
3. The pseudoscalars are Goldstone bosons and Goldstone theorem is satisfied,
4. The masses of vector and axial-vector bosons are obtained [10,11,15] and Weinberg's sum rule is satisfied,
5. Vector Meson Dominance is a natural result of this model and PCAC is satisfied,
6. The Lagrangian of the Wess-Zumino-Witten anomaly is the imaginary part of the effective Lagrangian,
7. Large-N_C expansion is realized in this effective theory. The tree diagrams are at the leading orders and loop

diagrams of mesons are at higher orders of large N_C expansion,

8. All the masses of mesons are below the cut-off. The theory is self-consistent,
9. There are five parameters: a universal coupling constant g, parameter m for quark condensate, and three current quark masses,
10. The masses and decay widths (strong, electromagnetic, and weak interactions) of the mesons are calculated. Form factors, $\pi\pi$ and πK scatterings [16] are studied. τ mesonic decays are systematic studied [12]. Theory agrees well with data.

The ChPT is the low-energy limit of any model of mesons. In this paper the low-energy behavior of the effective theory described by Lagrangian(2) is studied to see whether the effective theory of mesons goes back to ChPT at low energies and the 10 G-L coefficients of ChPT can be predicted.

In this effective theory [10, 11] meson resonances are involved. In order to compare with ChPT the momentum expansion to $O(p^4)$ is used. The quark masses and the G-L coefficients are scale-dependent quantities. In this paper the scale is the mass of the ρ-meson.

As a matter of fact, in ref. [10] two of the coefficients have been determined from the contribution of the ρ-resonance in $\pi\pi$ scattering. There are contact terms which make less contribution. In this paper a complete expressions of the coefficients $L_{1,2,3}$ are presented.

The vertex $\rho\pi\pi$ has been derived in ref. [10]

$$\mathcal{L}^{\rho\pi\pi} = f_{\rho\pi\pi}(q^2)\epsilon_{ijk}\rho^i_\mu\pi_j\partial^\mu\pi_k,$$
$$f_{\rho\pi\pi}(q^2) = \frac{2}{g}\{1+\frac{q^2}{2\pi^2 f_\pi^2}[(1-\frac{2c}{g})^2-4\pi^2c^2]\}, \tag{4}$$

where g is the universal coupling constant in this effective theory [10], the decay width of ρ-meson is calculated to be 146MeV which is in good agreement with the data, and q is the momentum of the ρ-meson. The effective Lagrangian of $\pi\pi$ scattering at low energy ($q^2 < m_\rho^2$) is derived from eq. (4) and eq. (13) of ref. [10]

$$\begin{aligned}\mathcal{L} &= \frac{1}{2f_\pi^2}\partial_\mu\pi^2\partial^\mu\pi^2 \\ &+\frac{16}{f_\pi^4}\{[\frac{1}{2}\frac{1}{(4\pi)^2}(1-\frac{2c}{g})^2(-1+\frac{4c}{g}+\frac{12c^2}{g^2})-\frac{c^4}{g^2}] \\ &\times\partial_\mu\pi_i\partial^\mu\pi_i\partial_\nu\pi_j\partial^\nu\pi_j \\ &+[\frac{1}{(4\pi)^2}(1-\frac{2c}{g})^2(1-\frac{4c}{g}-\frac{4c^2}{g^2})+\frac{c^4}{g^2}] \\ &\times\partial_\mu\pi_i\partial^\nu\pi_i\partial_\mu\pi_j\partial^\nu\pi_j\} \\ &-\frac{4}{f_\pi^4}(1-\frac{2c}{g})\frac{2c}{g}\{2gc+\frac{1}{\pi^2}(1-\frac{2c}{g})\} \\ &\times\{\partial_\mu\pi_i\partial^\mu\pi_i\partial_\nu\pi_j\partial^\nu\pi_j-\partial_\mu\pi_i\partial^\nu\pi_i\partial_\mu\pi_j\partial^\nu\pi_j\}. \end{aligned}\tag{5}$$

Comparing with Lagrangian of the CHPT(1), we obtain

$$\begin{aligned}2(L_1+L_2)+L_3 &= \frac{1}{4}\frac{1}{(4\pi)^2}(1-\frac{2c}{g})^4, \\ L_2 &= \frac{1}{4}\frac{c^4}{g^2}+\frac{1}{4}\frac{1}{(4\pi)^2}(1-\frac{2c}{g})^2(1-\frac{4c}{g}-\frac{4c^2}{g^2}) \\ &+\frac{1}{8}(1-\frac{2c}{g})\frac{c}{g}\{2gc+\frac{1}{\pi^2}(1-\frac{2c}{g})\}.\end{aligned}\tag{6}$$

Only two of the three coefficients are determined from $\pi\pi$ scattering. Therefore, another process is needed. We choose πK scattering [16, 17] at low energy to do the job. ρ and K* resonances contribute to πK scattering. Using eq. (13) in ref. [10], the $\rho\pi\pi$ vertex, and

$$\mathcal{L}^{K^*\pi\pi} = f_{\rho\pi\pi}(q^2)f_{abk}K^{a*}_\mu(K_b\partial^\mu\pi_k-\pi_k\partial^\mu K_b)\ \ [11], \tag{7}$$

in the chiral limit the two isospin amplitudes of πK scattering at low energy are derived as

$$\begin{aligned}T^{\frac{3}{2}} &= -\frac{2}{f_\pi^2}s+\frac{16}{f_\pi^4}\{\frac{1}{(4\pi)^2}\frac{4c^2}{g^2}(1-\frac{2c}{g})^2-\frac{c^4}{2g^2}\}\{t^2+u^2\} \\ &+\frac{16}{f_\pi^4}\{\frac{1}{(4\pi)^2}(1-\frac{2c}{g})^2(1-\frac{4c}{g}-\frac{4c^2}{g^2})+\frac{c^4}{g^2}\}s^2 \\ &-\frac{1}{f_\pi^4}\frac{4c}{g}(1-\frac{2c}{g})\{2gc+\frac{1}{\pi}(1-\frac{2c}{g})\} \\ &\times\{u(s-t)+t(s-u)\}, \\ T^{\frac{1}{2}} &= -\frac{1}{f_\pi^2}(3u-s)+\frac{16}{f_\pi^4}\{\frac{1}{(4\pi)^2}\frac{4c^2}{g^2}(1-\frac{2c}{g})^2-\frac{c^4}{2g^2}\}t^2 \\ &-\frac{8}{f_\pi^4}\{\frac{1}{(4\pi)^2}(1-\frac{2c}{g})^2(1-\frac{4c}{g}-\frac{16c^2}{g^2})+\frac{5}{2}\frac{c^4}{g^2}\}s^2 \\ &+\frac{8}{f_\pi^4}\{\frac{1}{(4\pi)^2}(1-\frac{2c}{g})^2(3-\frac{12c}{g}-\frac{16c^2}{g^2})+\frac{7}{2}\frac{c^4}{g^2}\}u^2 \\ &+\frac{1}{f_\pi^4}\frac{2c}{g}(1-\frac{2c}{g})\{2gc+\frac{1}{\pi}(1-\frac{2c}{g})\} \\ &\times\{-3s(u-t)+u(s-t)+4t(s-u)\},\end{aligned}\tag{8}$$

where

$$s=(p_1+p_2)^2,\quad t=(p_1-p_3)^2,\quad u=(p_1-p_4)^2,$$

$p_{1,3}$ are the momenta of initial and final pions and $p_{2,4}$ are the momenta of initial and final kaons, respectively. The two isospin amplitudes of πK scattering can be derived from the chiral perturbation theory(1) too. Comparing the partial waves obtained from eqs. (8) and CHPT, besides eqs. (6) obtained from $\pi\pi$ scattering we obtain

$$\begin{aligned}-2L_1+L_2 &= 0, \\ L_3 &= -\frac{3}{16}\frac{2c}{g}(1-\frac{2c}{g})\{2gc+\frac{1}{\pi^2}(1-\frac{2c}{g})\} \\ &-\frac{1}{2}\frac{1}{(4\pi)^2}\left(1-\frac{2c}{g}\right)^2(1-\frac{4c}{g}-\frac{8c^2}{g^2})-\frac{3}{4}\frac{c^4}{g^2}, \\ L_1 &= \frac{1}{32}\frac{2c}{g}(1-\frac{2c}{g})\{2gc+\frac{1}{\pi^2}(1-\frac{2c}{g})\} \\ &+\frac{1}{8}\frac{1}{(4\pi)^2}\left(1-\frac{2c}{g}\right)^2(1-\frac{4c}{g}-\frac{4c^2}{g^2})+\frac{1}{4}\frac{c^4}{g^2}.\end{aligned}\tag{9}$$

The first equation of eq. (9) has been obtained in ref. [8].

It is learned from ref. [10] that $g^2 \sim O(N_C)$, $f_\pi^2 \sim O(N_C)$, and $m_\rho^2 \sim O(1)$ in the large-N_C expansion. Therefore, $L_{1,2,3} \sim O(N_C)$. The predictions of L_{1-3} are made at the tree level, therefore, eq. (9) actually means that $-2L_1 + L_2 \sim O(1)$ in large-N_C expansion.

According to ref. [1b], the coefficients L_{4-8} are determined from the quark mass expansions of m_π^2, $m_{\rm K}^2$, m_η^2, f_π, $f_{\rm K}$, and f_η. Therefore, the quark mass term of the Lagrangian (5) is needed to be taken into account in deriving the effective Lagrangian of mesons. The expressions of the masses and the decay constants of the pseudoscalars are found from the real part of the effective Lagrangian. Using the method presented in ref. [10], in Euclidean space the real part of the effective Lagrangian of mesons with quark masses is written as

$$\begin{aligned}\mathcal{L}_{\rm RE} = \frac{1}{2}\int \mathrm{d}^D x \frac{\mathrm{d}^D p}{(2\pi)^D}\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{(p^2+m^2)^n} \\ \times \mathrm{Tr}\{(\gamma\cdot\partial - i\gamma\cdot v + i\gamma\cdot a\gamma_5)(\gamma\cdot\partial - i\gamma\cdot v - i\gamma\cdot a\gamma_5) \\ +2ip\cdot(\partial - iv - ia) + m\gamma\cdot Du - i[\gamma\cdot v, M] \\ +i\{\gamma\cdot a, M\}\gamma_5 - m(\hat{u}M + Mu) - M^2\}^n\,, \end{aligned} \quad (10)$$

where $D_\mu u = \partial_\mu u - i[v_\mu, u] + i\{a_\mu, u\}\gamma_5$ and $\hat{u} = \exp\{-i\gamma_5[\tau_i\pi_i + \lambda_a K_a + \lambda_8\eta_8 + \eta_0]\}$. Comparing with eq. (11) of ref. [10], there are new terms in which the quark mass matrix M is involved. As done in ref. [10], the mesons fields in eq. (10) have to be normalized.

The masses of the pseudoscalcar mesons are derived from eq. (10). The masses of the pseudoscalars can be calculated to any order in quark masses. Up to the second order in quark masses we obtain

$$\begin{aligned} m^2_{\pi^\pm} &= \frac{4}{f^2_{\pi^\pm}}\{-\frac{1}{3}\langle\bar{\psi}\psi\rangle(m_u+m_d) - \frac{F^2}{4}(m_u+m_d)^2\}, \\ m^2_{\pi^0} &= \frac{4}{f^2_{\pi^0}}\{-\frac{1}{3}\langle\bar{\psi}\psi\rangle(m_u+m_d) - \frac{F^2}{2}(m_u^2+m_d^2)\}, \\ m^2_{\rm K^+} &= \frac{4}{f^2_{\rm K^+}}\{-\frac{1}{3}\langle\bar{\psi}\psi\rangle(m_u+m_s) - \frac{F^2}{4}(m_u+m_s)^2\}, \\ m^2_{\rm K^0} &= \frac{4}{f^2_{\rm K^0}}\{-\frac{1}{3}\langle\bar{\psi}\psi\rangle(m_d+m_s) - \frac{F^2}{4}(m_d+m_s)^2\}, \\ m^2_{\eta_8} &= \frac{4}{f^2_{\eta_8}}\{-\frac{1}{3}\langle\bar{\psi}\psi\rangle\frac{1}{3}(m_u+m_d+4m_s) \\ &\quad -\frac{F^2}{6}(m_u^2+m_d^2+4m_s^2)\}. \end{aligned} \quad (11)$$

$\langle\bar{\psi}\psi\rangle$ is the quark condensate of three flavors. In terms of a cut-off Λ the quark condensate is expressed as

$$\begin{aligned}\langle\bar{\psi}\psi\rangle &= \frac{i}{(2\pi)^4}\mathrm{Tr}\int \mathrm{d}^4 p\frac{\gamma\cdot p - m\hat{u}}{p^2-m^2} \\ &= -\frac{3m^3 N_C}{4\pi^2}\{\frac{\Lambda^2}{m^2} - \log(\frac{\Lambda^2}{m^2}+1)\}. \end{aligned} \quad (12)$$

The universal coupling constant g is expressed as [10]

$$g^2 = \frac{F^2}{6m^2} = \frac{1}{2\pi^2}\{\log(\frac{\Lambda^2}{m^2}+1) + \frac{1}{\frac{\Lambda^2}{m^2}+1} - 1\}. \quad (13)$$

The decay constants of the pseudoscalars, f_π, $f_{\rm K}$, and f_η, are defined by normalizing the pseudoscalar fields as done in refs. [10, 11]. In this paper we calculate the contributions of the quark masses to these decay constants to $O(m_q)$. Using the effective Lagrangian (10), the decay constants can be calculated to any order in quark masses. As indicated in refs. [10, 11], there is mixing between the axial-vector field and corresponding pseudoscalar field. The mixing results in the shifting of the axial-vector field a_μ

$$a_\mu \to \frac{1}{g_a}a_\mu - \frac{c_a}{g_a'}\partial_\mu P, \quad (14)$$

where P is the corresponding pseudoscalar field, g_a is the normalization constant of the a_μ field, and c_a is the mixing coefficient. Both g_a and c_a are determined in the chiral limit in ref. [10]. It is obtained by eliminating the mixing between pion and a_1 fields that

$$\frac{c_a}{g_a'} = \frac{1}{g_a^2 m_a^2}\{\frac{F^2}{2} + (\frac{F^2}{8m^2} - \frac{3}{4\pi^2})2m(m_u+m_d)\}, \quad (15)$$

where m_a is the mass of the a_1-meson, which is determined from the Lagrangian (10) as

$$g_a^2 m_a^2 = F^2 + g^2 m_\rho^2 + 6g_{0a}^2 m(m_u+m_d), \quad (16)$$

where g_{0a}^2 is expressed as [10]

$$g_{0a}^2 = g^2(1 - \frac{1}{2\pi^2 g^2}). \quad (17)$$

Up to the first-order in quark masses, the decay constant f_π^2 is obtained from the Lagrangian (10)

$$f_\pi^2 = f_{\pi 0}^2\{1 + f\frac{m_u+m_d}{m}\}, \quad (18)$$

where

$$f_{\pi 0}^2 = F^2(1 - \frac{2c}{g}) \ \ [10] \quad (19)$$

and

$$\begin{aligned} f &= (1-\frac{2c}{g})(1-\frac{1}{2\pi^2 g^2}) - 1 \\ &\quad -\frac{4}{\pi^2 f_{\pi 0}^4}\frac{1}{3}\langle\bar{\psi}\psi\rangle m(1-\frac{2c}{g})(1-\frac{c}{g}). \end{aligned} \quad (20)$$

In the same way, $f^2_{\rm K^+}$, $f^2_{\rm K^0}$, and $f^2_{\eta_8}$ are found

$$f^2_{\rm K^+} = f^2_{\pi 0}\{1 + f\frac{m_u+m_s}{m}\}, \quad (21)$$

$$f^2_{\rm K^0} = f^2_{\pi 0}\{1 + f\frac{m_d+m_s}{m}\}, \quad (22)$$

$$f^2_{\eta_8} = f^2_{\pi 0}\{1 + \frac{1}{3}f\frac{m_u+m_d+4m_s}{m}\}. \quad (23)$$

Substituting eqs. (18,21,22,23) into eq. (11), to the second order in quark masses the masses of the pseudoscalar mesons are obtained

$$\begin{aligned}
m^2_{\pi^\pm} &= \frac{4}{f^2_{\pi 0}}\{-\frac{1}{3}\langle\bar{\psi}\psi\rangle(m_u+m_d) \\
&\quad -\frac{F^2}{4}(m_u+m_d)^2+\frac{f}{3}\langle\bar{\psi}\psi\rangle\frac{1}{m}(m_u+m_d)^2\}, \\
m^2_{\pi^0} &= \frac{4}{f^2_{\pi 0}}\{-\frac{1}{3}\langle\bar{\psi}\psi\rangle(m_u+m_d) \\
&\quad -\frac{F^2}{2}(m^2_u+m^2_d)+\frac{f}{3}\langle\bar{\psi}\psi\rangle\frac{1}{m}(m_u+m_d)^2\}, \\
m^2_{K^+} &= \frac{4}{f^2_{\pi 0}}\{-\frac{1}{3}\langle\bar{\psi}\psi\rangle(m_u+m_s) \\
&\quad -\frac{F^2}{4}(m_u+m_s)^2+\frac{f}{3}\langle\bar{\psi}\psi\rangle\frac{1}{m}(m_u+m_s)^2\}, \\
m^2_{K^0} &= \frac{4}{f^2_{\pi 0}}\{-\frac{1}{3}\langle\bar{\psi}\psi\rangle(m_d+m_s) \\
&\quad -\frac{F^2}{4}(m_d+m_s)^2+\frac{f}{3}\langle\bar{\psi}\psi\rangle\frac{1}{m}(m_d+m_s)^2\}, \\
m^2_{\eta_8} &= \frac{4}{f^2_{\pi 0}}\{-\frac{1}{3}\langle\bar{\psi}\psi\rangle\frac{1}{3}(m_u+m_d+4m_s) \\
&\quad -\frac{F^2}{4}\frac{2}{3}(m^2_u+m^2_d+4m^2_s) \\
&\quad +\frac{1}{3}f\langle\bar{\psi}\psi\rangle\frac{1}{9}\frac{1}{m}(m_u+m_d+4m_s)^2\} \\
&= \frac{4}{f^2_{\pi 0}}\{-\frac{1}{3}\langle\bar{\psi}\psi\rangle\frac{1}{3}(m_u+m_d+4m_s) \\
&\quad +(\frac{f}{m}\frac{1}{3}\langle\bar{\psi}\psi\rangle-\frac{F^2}{4})\frac{1}{9}(m_u+m_d+4m_s)^2 \\
&\quad -\frac{2F^2}{9}[\frac{1}{2}(m_u+m_d)-m_s]^2-\frac{F^2}{12}(m_d-m_u)^2\}. \quad (24)
\end{aligned}$$

The mass formulas (24) are the same as the ones obtained by ChPT([1b]). The mass difference of charged and neutral pions is found from eqs. (24)

$$m^2_{\pi^\pm}-m^2_{\pi^0}=(1-\frac{2c}{g})^{-1}(m_d-m_u)^2. \quad (25)$$

Comparing eqs. (24) with eqs. (10.7,10.8) of ref. [1](b)), the following coefficients of the ChPT are predicted:

$$L_4=0, \qquad L_6=0, \quad (26)$$

$$L_5=\frac{f^2_{\pi 0}f}{8mB_0}, \quad (27)$$

$$L_8=-\frac{F^2}{16B^2_0}, \quad (28)$$

$$3L_7+L_8=-\frac{F^2}{16B^2_0}, \qquad L_7=0, \quad (29)$$

where

$$B_0=\frac{4}{f^2_{\pi 0}}(-\frac{1}{3})\langle\bar{\psi}\psi\rangle. \quad (30)$$

Using eq. (30), L_5 and L_8 are written as

$$\begin{aligned}
L_5 &= \frac{1}{32Q}(1-\frac{2c}{g})\{(1-\frac{2c}{g})^2(1-\frac{1}{2\pi^2g^2}) \\
&\quad -(1-\frac{2c}{g})+\frac{4}{\pi^2}Q(1-\frac{c}{g})\}, \quad (31) \\
L_8 &= -\frac{1}{1536g^2Q^2}(1-\frac{2c}{g})^2, \quad (32)
\end{aligned}$$

where

$$Q=-\frac{1}{108g^4}\frac{1}{m^3}\langle\bar{\psi}\psi\rangle. \quad (33)$$

Equations (12,13) show that Q is a function of the universal coupling constant g only and is determined to be 4.54.

Both L_5 and L_8 are at $O(N_C)$. L_4, L_6, and L_7 are from loop diagrams of mesons and are at $O(1)$ in the large-N_C expansion.

According to refs. [1b,3], L_9 and L_{10} are determined by $\langle r^2\rangle_\pi$ and the amplitudes of pion radiative decay, $\pi^-\to e^-\gamma\nu$

$$L_9=\frac{f^2_\pi}{48}\langle r^2_\pi\rangle, \quad (34)$$

$$L_9=\frac{1}{32\pi^2}\frac{R}{F^V}, \quad (35)$$

$$L_{10}=\frac{1}{32\pi^2}\frac{F^A}{F^V}-L_9, \quad (36)$$

where R, F^V, and F^A are r_A, h_V,and h_A of ref. [3], respectively. In ref. [12] we have derived the expression of pion radius as

$$\langle r^2\rangle_\pi=\frac{6}{m^2_\rho}+\frac{3}{\pi^2f^2_\pi}\{(1-\frac{2c}{g})^2-4\pi^2c^2\}, \quad (37)$$

which agrees with the data very well. Using eqs. (34,37), it is predicted

$$L_9=\frac{1}{4}cg+\frac{1}{16\pi^2}\{(1-\frac{2c}{g})^2-4\pi^2c^2\}, \quad (38)$$

On the other hand, L_9 is determined by eq. (35) too. The three form factors of the decay amplitude of $\pi^-\to e^-\gamma\nu$ are presented in our paper [12]

$$F^V=\frac{m_\pi}{2\sqrt{2}\pi^2f_\pi}, \quad (39)$$

$$F^A=\frac{1}{2\sqrt{2}\pi^2}\frac{m_\pi}{f_\pi}\frac{m^2_\rho}{m^2_a}(1-\frac{2c}{g})(1-\frac{1}{2\pi^2g^2})^{-1}, \quad (40)$$

$$\begin{aligned}
R &= \frac{g^2}{\sqrt{2}}\frac{m_\pi}{f_\pi}\frac{m^2_\rho}{m^2_a}\{\frac{2c}{g}+\frac{1}{\pi^2g^2}(1-\frac{2c}{g})\} \\
&\quad \times(1-\frac{1}{2\pi^2g^2})^{-1}+\sqrt{2}cg\frac{m_\pi}{f_\pi} \quad [14]. \quad (41)
\end{aligned}$$

Using the mass formula of the a_1-meson (in the chiral limit) [10]

$$(1-\frac{1}{2\pi^2g^2})m^2_a=\frac{F^2}{g^2}+m^2_\rho,$$

Table 3. Predictions of the Values of the coefficients

10^3L_1	10^3L_2	10^3L_3	10^3L_4	10^3L_5	10^3L_6	10^3L_7	10^3L_8	10^3L_9	10^3L_{10}
1.0	2.0	−5.16	0	4.77	0	0	−0.079	8.3	−7.1

it is obtained

$$\frac{m_\rho^2}{m_a^2}(1-\frac{1}{2\pi^2g^2})^{-1}=1-\frac{2c}{g}. \tag{42}$$

Substituting eq. (42) into R (41), it is derived

$$R=\frac{1}{3\sqrt{2}}m_\pi f_\pi\langle r^2\rangle_\pi. \tag{43}$$

This is the expression obtained by applying PCAC to this process [18]. Therefore, the coefficient L_9 derived from eq. (35) is the same as the one obtained from eq. (34).

Using eq. (42), the ratio $\frac{F^A}{F^V}$ is written as

$$\frac{F^A}{F^V}=(1-\frac{2c}{g})^2. \tag{44}$$

The coefficient L_{10} is found from eq. (36)

$$L_{10}=-\frac{1}{4}cg+\frac{1}{4}c^2-\frac{1}{32\pi^2}(1-\frac{2c}{g})^2. \tag{45}$$

Both L_9 and L_{10} are at $O(N_C)$.

In the expressions of the coefficients (6,9,26,29,31, 32,38,45) there are two parameters: g and f_π^2/m_ρ^2 which have been determined already. The coefficients of the ChPT are, completely, predicted by this effective chiral theory and their numerical values are listed in table 3.

Some of the G-L coefficients have recently re-evaluated on the basis of an $O(p^6)$ analysis [19]. Table 1 shows that there are changes in $L_{2,5,8}$. Before comparing our theoretical predictions of the G-L coefficients with the phenomenological values it is necessary to point out that

1. at low energies the effective chiral theory of mesons goes back to the ChPT,
2. up to $O(m_q^2)$ the mass formulas of the pseudoscalars (24) obtained from this effective theory are the same as the ones in the ChPT,
3. up to $O(m_q)$ the expressions of f_π, $f_{\rm K}$, and f_η(18-23) are the same as in the ChPT,
4. the form factor of pion [20], $\pi\pi$ and πK scatterings, form factors of $\pi\to e\gamma\nu$ have been studied by this effective theory. Theoretical results agree well with data. The values of these physical quantities at low energies are used to determine $L_{1,2,3,9,10}$.

The theoretical values of $L_{1,2,3}$ are compatible with their phenomenological values shown in table 1. However, there are differences. In ref. [9,19] the values of $L_{1,2,3}$ are determined by fitting the data of $\pi\pi$ scattering and $\rm K_{e4}$. In this paper we use $\pi\pi$ and πK to determine them. $\rm K_{l4}$ decays have been studied by using this effective theory [21].

To the leading order in N_C $L_{4,6}=0(26)$ are obtained and are the same as determined by Zweig rule [9,19]. L_7 is compatible with the phenomenological value. The value of L_5 is much greater than the phenomenological value which is determined by the ratio of $\frac{f_{\rm K}}{f_\pi}$. The expressions of both $f_{{\rm K},\pi}$(18,21,22) are the same as in ref. [1]. In this paper a greater value of the quark condensate(12) is obtained by input $g=0.39$ which is determined by fitting $\rho\to{\rm ee}^+$

$$\langle\bar{u}u\rangle=\langle\bar{d}d\rangle=\langle\bar{s}s\rangle=-(0.378\,{\rm GeV})^3. \tag{46}$$

Equation (31) shows that a greater quark condensate leads to greater value of L_5. On the other hand, $\frac{f_{\rm K}}{f_\pi}=1.19$ is obtained in this paper, which is 2.5% away from the data. Because of the same reason a smaller L_8 (32) is obtained in this paper.

The theoretical values of $L_{9,10}$ are compatible withe their phenomenological values.

To summarize the results, the effective chiral theory of mesons goes back to ChPT at low energies. The orders of the G-L coefficients of the ChPT in the large-N_C expansion are predicted as $L_{1,2,3,5,8,9,10}\sim O(N_C)$ and $L_{4,6}$ are $O(1)$. These predictions agree with ref. [1b]. It is also predicted that $L_7\sim O(1)$ in this paper. There is no adjustable parameter in predicting the G-L coefficients of the ChPT.

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